A Simple Deviation from Relativity

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A test of special relativity is proposed by conceiving a rather natural generalization of the minkowskian spacetime. This is mathematically similar to generalizing the notion of finite dimensional Banach space of the related Hilbert space concept. A corresponding experiment might be feasible with appropriate quantum optical methods.

I. Introduction: A Simple Relativistic Particle

This is known to obey the Hamiltonian principle of motion

$$\delta \int_{x^0}^{x^1} \mathrm{d}\tau = 0 \quad \text{with} \quad \delta x^0 = 0 = \delta x^1 , \tag{1}$$

where $\mathrm{d}\tau := \sqrt{\mathrm{d}t^2 - c_0^{-2}\,\mathrm{d}\boldsymbol{r}^2} = \mathrm{d}t\sqrt{1 - c_0^{-2}\,\boldsymbol{v}^2}$ is the eigentime increment, $v := \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ the usual velocity of the particle, and c_0 the velocity of light in vacuo. Hence the Lagrangean L is given by

$$L(\boldsymbol{v}) := -c_0^2 m\sqrt{1 - c_0^{-2}\,\boldsymbol{v}^2} = -c_0^2 m + (1/2)\,m\,\boldsymbol{v}^2 + \dots . \tag{2}$$

The corresponding canonical momentum $\boldsymbol{p} := \partial L/\partial \boldsymbol{v}$ turns out to be

$$\boldsymbol{p} = m\,\boldsymbol{v}/\sqrt{1 - c_0^{-2}\,\boldsymbol{v}^2} ,$$

so that

$$\boldsymbol{v} = c_0\,\boldsymbol{p}/\sqrt{c_0^2 m^2 + \boldsymbol{p}^2} ; \tag{3}$$

and thence the Hamiltonian H, as defined by $H(\boldsymbol{p}) := \boldsymbol{p}\cdot\boldsymbol{v}(\boldsymbol{p}) - L(\boldsymbol{v}(\boldsymbol{p}))$, becomes

$$H(\boldsymbol{p}) = c_0\sqrt{c_0^2 m^2 + \boldsymbol{p}^2} . \tag{4}$$

With the canonical energy $\eta := H$ we check the well known square of the mass invariant: $c_0^{-2}\eta^2 - \boldsymbol{p}^2 = c_0^2 m^2$. Hamilton's canonical equations $\mathrm{d}\boldsymbol{p}/\mathrm{d}t = -\partial H/\partial \boldsymbol{r}$, $\mathrm{d}\boldsymbol{r}/\mathrm{d}t = +\partial H/\partial \boldsymbol{p}$ here read $\mathrm{d}\boldsymbol{p}/\mathrm{d}t = 0$ and $\mathrm{d}\boldsymbol{r}/\mathrm{d}t = c_0\,\boldsymbol{p}/\sqrt{c_0^2 m^2 + \boldsymbol{p}^2}$. The last quantity coincides, of course, with the velocity $\boldsymbol{v}$ in (3); but now we know that the momentum is a constant motion: $\boldsymbol{p}(t) = \boldsymbol{p}_0$. Consequently $\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0\,t$ with $\boldsymbol{v}_0 = c_0\boldsymbol{p}_0/\sqrt{c_0^2 m^2 + \boldsymbol{p}_0^2}$, so that $\boldsymbol{p}_0 = m\,\boldsymbol{v}_0/\sqrt{1 - c_0^{-2}\,\boldsymbol{v}_0^2}$. This is free motion within relativistic kinematics: $|\boldsymbol{v}_0| < c_0$ as $m^2 > 0$.

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II. Introduction of a Weak Ether Wind for this Mass Point

We replace the extremely simple expression $\mathrm{d}\tau^2 = \sum_\mu \mathrm{d}x_\mu\,\mathrm{d}x^\mu$ in the variational principle by the still rather simple formula

$$\mathrm{d}\tau_\sigma^2 = \mathrm{d}\tau^2 + \sigma\,c_0^{-4}\Big(\sum_\mu w_\mu\,\mathrm{d}x^\mu\Big)^2 \quad \text{with} \quad |\sigma| \ll 1 , \tag{5}$$

where the w_μ are the covariant components of an external four-velocity w, and $u^\mu := \mathrm{d}x^\mu/\mathrm{d}t$ the contravariant components of the particle four-velocity u, so that $(u^0; u^1, u^2, u^3) = (c_0; v_x, v_y, v_z)/\sqrt{1 - c_0^{-2}\boldsymbol{v}^2}$ with $\boldsymbol{v} = (v_x, v_y, v_z)$. Both four-velocities are unit-vectors times c_0, which means that $w^2 = c_0^2$ and $u^2 = c_0^2$, where $w^2 := w\cdot w := \sum_\mu w_\mu\,w^\mu := w_0^2 - \boldsymbol{w}^2$ as well as $u^2 := u\cdot u := \sum_\mu u_\mu\,u^\mu := u_0^2 - \boldsymbol{u}^2$. We are using the usual Ricci calculus: $a\cdot b := \sum_\mu a^\mu\,b_\mu$ and $b_\mu = \sum_\nu g_{\mu\nu}\,b^\nu$, where $g_{\mu\nu} := g_\mu\,\delta_{\mu\nu}$ with $-g_0 = g_1 = g_2 = g_3 = -1$. Thus our notions and notations are formally covariant; yet the physical structure is not relativistically invariant, because w distinguishes one inertial frame of reference from all the others. This "absolutely resting" frame is that one in which the only nonvanishing component of w is the temporal, hence

$$\overset{0}{w} = (c_0; 0, 0, 0) \tag{6}$$

in the preferred frame of reference, and only in this frame.

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The ether we are considering is not a mechanical one but like that of H. A. Lorentz, of a purely functional nature. In that it is similar to the accelerational or inertial ether that Einstein and Hermann Weyl conceived for gravitation. But for now, we disregard gravity.

It is important to note that $(x^0; x^1, x^2, x^3) := (c_0 t; x, y, z)$ are the minkowskian coordinates of the event x as occurring in the original *field equations* of electromagnetism. This part of the theory is assumed to be *exactly Lorentz-invariant*. The particle, on the other hand, is supposed to be subject to an indefinite metric which is a pseudo-banachian (or pseudo-finslerian) generalization of the usual pseudo-euclidean (or pseudo-riemannian) type of metric.

III. The Canonical Formalism

For the *Lagrangean* of the particle we thus obtain the following expression, which no longer obeys strict relativity:

$$L(\boldsymbol{v}) := -c_0^2 m \sqrt{1 - c_0^{-2} \boldsymbol{v}^2 + \sigma c_0^{-4} (w_0 c_0 - \boldsymbol{w} \cdot \boldsymbol{v})^2} \quad (7)$$

with $w_0 = c_0$ and $\boldsymbol{w} = \boldsymbol{0}$ in the resting frame, according to (6). The corresponding values of w_0 and $\boldsymbol{w}$ in a moving frame are given by the appropriate Lorentz transformation to the effect

$$\overset{1}{w} = (c_0; 0, 0, c_0 \beta) / \sqrt{1 - \beta^2}\,, \quad (8)$$

if $(0, 0, c_0 \beta)$ is the absolute velocity of the moving frame, where "absolute" means "relative to the resting frame". The four components in (6) are absolute ones, those in (8) being relative.

The *canonical momenta* are again easily evaluated with the following result:

$$\boldsymbol{p} = m \frac{\boldsymbol{v} + \sigma c_0^{-2} (w_0 c_0 - \boldsymbol{w} \cdot \boldsymbol{v})\, \boldsymbol{w}}{\sqrt{1 - c_0^{-2} \boldsymbol{v}^2 + \sigma c_0^{-4} (w_0 c_0 - \boldsymbol{w} \cdot \boldsymbol{v})^2}}\,, \quad (9)$$

which agrees to the first order in σ with the expression

$$\boldsymbol{p} = m \left[\frac{\boldsymbol{v} + \sigma c_0^{-2} (w_0 c_0 - \boldsymbol{w} \cdot \boldsymbol{v}\, \boldsymbol{w})}{\sqrt{1 - c_0^{-2} \boldsymbol{v}^2}} - \frac{1}{2} \sigma c_0^{-4} \frac{(w_0 c_0 - \boldsymbol{w} \cdot \boldsymbol{v}\, \boldsymbol{v})^2}{\sqrt{1 - c_0^{-2} \boldsymbol{v}^2}^{\,3}} \right]. \quad (10)$$

In order to solve this for $\boldsymbol{v}$ we first recall the zeroth-order relations (3), so that $w_0 c_0 - \boldsymbol{w} \cdot \boldsymbol{v} = c_0 (w_0 - \boldsymbol{w} \cdot \boldsymbol{p} / \sqrt{c_0^2 m^2 + \boldsymbol{p}^2})$ to the zeroth order of σ. We then derive the first-oder subsidiary relations

$$\begin{aligned}(c_0^2 m^2 + \boldsymbol{p}^2)(1 - c_0^{-2} \boldsymbol{v}^2) &= c_0^2 m^2 + \sigma m^2 \left[2 \frac{w_0 c_0 - \boldsymbol{w} \cdot \boldsymbol{v}}{c_0^2} \boldsymbol{w} \cdot \boldsymbol{v} - \frac{(w_0 c_0 - \boldsymbol{w} \cdot \boldsymbol{v})^2}{c_0^4} \frac{\boldsymbol{v}^2}{1 - c_0^{-2} \boldsymbol{v}^2} \right] \\ &= c_0^2 m^2 - \sigma c_0^{-2} \left(w_0 - \frac{\boldsymbol{p} \cdot \boldsymbol{w}}{\sqrt{c_0^2 m^2 + \boldsymbol{p}^2}} \right) \cdot \left(w_0 \boldsymbol{p}^2 - \frac{2 c_0^2 m^2 + \boldsymbol{p}^2}{\sqrt{c_0^2 m^2 + \boldsymbol{p}^2}} \boldsymbol{p} \cdot \boldsymbol{w} \right). \end{aligned} \quad (11)$$

With their help we find that

$$\boldsymbol{v} = \frac{c_0 \boldsymbol{p}}{\sqrt{c_0^2 m^2 + \boldsymbol{p}^2}} + \sigma c_0^{-1} \left(w_0 - \frac{\boldsymbol{w} \cdot \boldsymbol{p}}{\sqrt{c_0^2 m^2 + \boldsymbol{p}^2}} \right) \cdot \left[\frac{1}{2} \left(w_0 + \frac{\boldsymbol{p} \cdot \boldsymbol{w}}{\sqrt{c_0^2 m^2 + \boldsymbol{p}^2}} \right) \frac{\boldsymbol{p}}{\sqrt{c_0^2 m^2 + \boldsymbol{p}^2}} - \boldsymbol{w} \right] \quad (12)$$

to the first σ-order. By inserting this into the definition of the Hamiltonian we finally obtain

$$H(\boldsymbol{p}) = c_0 \sqrt{c_0^2 m^2 + \boldsymbol{p}^2} \cdot \left[1 + \tfrac{1}{2} \sigma c_0^{-2} \left(w_0 - \frac{\boldsymbol{p} \cdot \boldsymbol{w}}{\sqrt{c_0^2 m^2 + \boldsymbol{p}^2}} \right)^2 \right] \quad (13)$$

up to first order in σ inclusive. By using the *imperturbed energy*

$$\eta := c_0 \sqrt{c_0^2 m^2 + \boldsymbol{p}^2}\,, \quad (14)$$

this reads $H = \eta + \frac{1}{2} \sigma \eta^{-1} (w \cdot p)^2$, where the (formal invariant or) four-scalar $w \cdot p := c_0^{-2} w_0 \eta - \boldsymbol{w} \cdot \boldsymbol{p}$ is used. A check is provided by comparing the Hamilton equation $\boldsymbol{v} = \partial H(\boldsymbol{p}) / \partial \boldsymbol{p}$ with (12).

IV. The Kinematically Perturbed Atom

Next we apply this hypothesis to a *hydrogen atom* by assuming an infinitely heavy spinless point proton moving with the constant velocity $c_0 \beta$, and a spinless point electron which – consequently – has also no magnetic moment. This means that the Coulomb energy $-(e^2 / 4 \pi \varepsilon_0) |\boldsymbol{r}|^{-1}$ has to be subtracted from the Lagrangean and thence added to the Hamiltonian.

This now reads

$$H(\boldsymbol{r},\boldsymbol{p}) = c_0 \sqrt{c_0^2 m_e^2 + \boldsymbol{p}^2} \cdot \left[1 + \tfrac{1}{2}\sigma c_0^{-2}\left(w_0 - \frac{\boldsymbol{p}\cdot\boldsymbol{w}}{\sqrt{c_0^2 m_e^2 + \boldsymbol{p}^2}}\right)^2\right] - \frac{\pi_0}{4\pi}\frac{e^2}{|\boldsymbol{r}|}, \tag{15}$$

where $\pi_0 := \varepsilon_0^{-1}$, and $(w_0, \boldsymbol{w})$ is given by (8). We may assume that $|\boldsymbol{p}| \ll c_0 m_e$; in the n-th level of the Rydberg atom, $c_0^{-1} m_e^{-1} |\boldsymbol{p}| \approx \alpha/n$, where $\alpha := c_0^{-1}\hbar^{-1} e^2 \pi_0/4\pi = 1/137.0(4)$ and $n \simeq \frac{1}{2}\cdot 10^2$. Therefore

$$H(\boldsymbol{r},\boldsymbol{p}) = c_0^2 m_e + \tfrac{1}{2} m_e^{-1}\boldsymbol{p}^2 + \left[\tfrac{1}{2} c_0^{-2} w_0^2 (c_0^2 m_e + \tfrac{1}{2} m_e^{-1}\boldsymbol{p}^2) - c_0^{-1} w_0 \boldsymbol{p}\cdot\boldsymbol{w} + \tfrac{1}{2} c_0^{-2} m_e^{-1} (\boldsymbol{p}\cdot\boldsymbol{w})^2\right] - \frac{\pi_0}{4\pi}\frac{1}{|\boldsymbol{r}|} \tag{16}$$

in the first order of σ and in the second order of $c_0^{-1}|\boldsymbol{v}|$. Again we may check that $\boldsymbol{v} = \partial H/\partial \boldsymbol{p}$, where

$$\boldsymbol{v} = m_e^{-1}\boldsymbol{p} + \sigma\left(\tfrac{1}{2} c_0^{-2} m_e^{-1} w_0^2 \boldsymbol{p} - c_0^{-1} w_0 \boldsymbol{w} + c_0^{-2} m_e^{-1} \boldsymbol{p}\cdot\boldsymbol{w} \otimes \boldsymbol{w}\right) \tag{17}$$

for $\sigma \ll 1$ and $|\boldsymbol{v}| \ll c_0$, in the first order of both. The rest energy of the electron accounts to $c_0^2 m_e = 0.510.99(9)$ MeV.

All our relations are valid in the quantal variant of this particle dynamics as well as in its classical version. The quantization of the originally classical reading is as usually done by straightforwardly interpreting all dynamical variables as linear operators which act – according to Schrödinger and Born – on a probability amplitude $\psi(\boldsymbol{r})$. The position $\boldsymbol{r}$ is then a multiplicative operator and the momentum $\boldsymbol{p} = -i\hbar\,\partial/\partial\boldsymbol{r}$ a differential operator, derived from the de Broglie relation $\boldsymbol{p} = \hbar\boldsymbol{k}$ together with Fouriers relation $\boldsymbol{k} = -i\,\partial/\partial\boldsymbol{r}$.

As is well known, the stationary states of our perturbed hydrogen atom are represented by the normalizable eigenfunctions ψ_{nlm} of the Hamilton operator $H(\boldsymbol{r}, -i\hbar\,\partial/\partial\boldsymbol{r})$, belonging to negative eigenvalues E_{nlm} with $n = 1, 2, 3, \ldots$; $l = 0, 1, 2, \ldots, n-1$; and $m = l, l-1, \ldots, 1-l, -l$. For $\sigma = 0$ the eigenfunctions are of the form $\mathring{\psi}_{nlm}(\boldsymbol{r}) = \mathscr{R}_{nl}(r)\mathscr{Y}_{lm}(\vartheta, \phi)$; and their energies $\mathring{E}_n = -\frac{1}{2}(\pi_0/4\pi)^2 e^4 \hbar^{-2} m_e/n^2$ depend only on $n := 1 + n_r + l$, where $n_r = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots$ may be freely chosen. Now it will be especially convenient to have the z-axis parallel to our geometric perturbation field $\boldsymbol{w}$, as we have already assumed in (8). The ensuing calculations are simplified by the symmetry relation $\langle \boldsymbol{p}\rangle^{(0)}_{nlm} = 0$ and the virial theorem $\frac{1}{2} m^{-1}\langle \boldsymbol{p}^2\rangle^{(0)}_n = -\mathring{E}_n$. Finally we have $\langle(\boldsymbol{p}\cdot\boldsymbol{w})^2\rangle^{(0)}_{nlm} = \langle \boldsymbol{p}^2\rangle^{(0)}_n \langle\cos^2\vartheta\rangle^{(0)}_{lm}$ with $\langle\cos^2\vartheta\rangle^{(0)}_{lm} = m^2/l(l+1)$ according to the Wigner-Eckart theorem *. Hence

$$E_{nlm} = -\frac{1}{2}\left(\frac{\pi_0 e^2}{4\pi}\right)^2 \frac{\hbar^{-2} m_e}{n^2} \cdot \left\{1 - \sigma\left[c_0^{-2} w_0^2 + c_0^{-2} \boldsymbol{w}^2 \frac{m^2}{l(l+1)}\right]\right\} + \tfrac{1}{2}\sigma m_e w_0^2. \tag{18}$$

For fast atoms we have approximately $c_0^{-2}\boldsymbol{w}^2 = 1$. Therefore the spread of the sublevel E_{nl} becomes

$$\delta E_{nl} = E_{nll} - E_{nl0} = \tfrac{1}{2}\sigma\left(\frac{\pi_0 e^2}{4\pi}\right)^2 \frac{\hbar^{-2} m_e}{n^2}\,\frac{\boldsymbol{w}^2}{c_0^2}\,\frac{l}{l+1}. \tag{19}$$

Its maximum value is

$$\delta E_n = \delta E_{n(n-1)} = \tfrac{1}{2}\sigma\left(\frac{\pi_0 e^2}{4\pi}\right)^2 \frac{\hbar^{-2} m_e}{n^2} = \sigma\cdot|\mathring{E}_n| = \sigma\cdot\frac{\hbar c_0 R_e}{n^2} \tag{20}$$

for $n \gg 1$ and $|\boldsymbol{w}| \approx c_0$, which is equivalent to $0 < 1 - \beta \ll 1$. Here, $\hbar c_0 R_e = \frac{1}{2}\alpha^2 c_0^2 m_e = 13.60(6)$ eV is the Rydberg energy, $R_e/2\pi = \frac{1}{2} e^4 (4\pi\varepsilon_0)^{-2} c_0^{-1}\hbar^{-3}/2\pi = 1.097'373'15(7)\cdot 10^7\ \mathrm{m}^{-1}$ being the inverse Rydberg wave length.

V. Contact with Experiment

With present laser techniques ** it seems possible to achieve an accuracy of about $|\sigma| < 10^{-6}$ for a test of this elementary kind of ether wind.

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* Because of $\sum_{-l}^{m=l} m^2 = \frac{1}{3} l(l+1)(2l+1)$ we obtain

$$\sum_{-l}^{m=l} \langle\cos^2\vartheta\rangle^{(0)}_{lm} = \tfrac{1}{3}(2l+1),$$

hence

$$\sum_{-l}^{m=l} \langle(\boldsymbol{p}\cdot\boldsymbol{w})^2\rangle^{(0)}_{nlm} = \tfrac{1}{3}(2l+1)\,\boldsymbol{w}^2\langle\boldsymbol{p}^2\rangle^{(0)}_n,$$

as must be by symmetry.

** H. C. Bryant, Private communication (1987).